Uniform coverings of 2-paths in the complete graph and the complete bipartite graph

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Abstract Let G be a graph and H a subgraph of G. A $D(G,H,\lambda)$ *design* is a collection $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every 2-path (path of length 2) in G lies in exactly λ subgraphs in $\mathcal{D}$. The problem of constructing $D(K_n, C_n, 1)$ designs is the so-called Dudeney's round table problem. We denote by C_k a cycle on k vertices and by P_k a path on k vertices.

In this paper, we construct $D(K_{n,n}, C_{2n}, 1)$ designs and $D(K_n, P_n, 1)$ designs when $n \equiv 0, 1, 3 \pmod 4$; and $D(K_{n,n}, C_{2n}, 2)$ designs and $D(K_n, P_n, 2)$ designs when $n \equiv 2 \pmod 4$. The existence problems of $D(K_{n,n}, C_{2n}, 1)$ designs and $D(K_n, P_n, 1)$ designs for $n \equiv 2 \pmod 4$ remain open.

1 Introduction

Consider a graph G and a subgraph H of G. A $D(G,H,\lambda)$ *design* is a collection $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every 2-path (path of length 2) in G lies in exactly λ subgraphs in $\mathcal{D}$. We call this design a *Dudeney design*. A $D(G,H,\lambda)$ design is *resolvable* or *vertex-resolvable* if the subgraphs in the design can be partitioned into classes so that every vertex appears exactly once in each class. Each such class is called a *parallel class* of the design.

Let K_n be the complete graph on n vertices, $K_{n,n}$ the complete bipartite graph on the partite sets with n vertices each, C_k a cycle on k vertices, and P_k a path on k vertices. We restrict our attention to $D(G,H,\lambda)$ designs in which G is K_n or $K_{n,n}$ and H is C_k or P_k.

The problem of constructing $D(G,H,\lambda)$ designs have been solved for the following cases:

1. $D(K_n, P_3, \lambda)$ designs (trivial) and resolvable $D(K_n, P_3, \lambda)$ designs ([4] Th. 2.9)
2. $D(K_n, C_3, \lambda)$ designs (trivial) and resolvable $D(K_n, C_3, \lambda)$ designs ([4] Th. 2.9)
3. $D(K_n, P_4, 1)$ designs ([4] Th. 2.20)
4. $D(K_n, C_4, \lambda)$ designs [5] and resolvable $D(K_n, C_4, 1)$ designs [9]
5. $D(K_n, P_5, 1)$ designs [10]

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6. $D(K_n, P_6, 1)$ designs [12, 13]

7. $D(K_n, C_6, 1)$ designs [11]

8. $D(K_n, P_7, 1)$ designs [1]

9. $D(K_{n,n}, P_4, 1)$ designs and resolvable $D(K_{n,n}, P_4, 1)$ designs ([4] Th. 3.3)

10. $D(K_{n,n}, C_4, 1)$ designs and resolvable $D(K_{n,n}, C_4, 1)$ designs ([4] Th. 3.1).

A $D(K_n, C_n, 1)$ design is a solution of the famous *Dudeney's round table problem* which asks for a seating of n people at a round table on consecutive days so that each person sat between every pair of other people exactly once [2, 3]. It has been conjectured[2] that there exists a $D(K_n, C_n, 1)$ design for every n, but it has not been solved in general [7]. The following theorem is known.

Theorem A [6, 8] *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n, C_n, 1)$ design when n is even.*
(2) *There exists a $D(K_n, C_n, 2)$ design when n is odd.*

In this paper, we consider $D(K_{n,n}, C_{2n}, \lambda)$ designs and $D(K_n, P_n, \lambda)$ designs. We may call the problem of constructing $D(K_n, P_n, 1)$ designs for every n *Dudeney's counter table problem.* We obtain the following results.

Theorem 1.1 *Let $n \geq 2$ be an integer.*
(1) *There exists a $D(K_{n,n}, C_{2n}, 1)$ design when n is odd.*
(2) *There exists a $D(K_{n,n}, C_{2n}, 1)$ design when $n \equiv 0 \pmod 4$.*
(3) *There exists a $D(K_{n,n}, C_{2n}, 2)$ design when $n \equiv 2 \pmod 4$.*

Theorem 1.2 *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n, P_n, 1)$ design when n is odd.*
(2) *There exists a $D(K_n, P_n, 1)$ design when $n \equiv 0 \pmod 4$.*
(3) *There exists a $D(K_n, P_n, 2)$ design when $n \equiv 2 \pmod 4$.*

The existence problems of $D(K_{n,n}, C_{2n}, 1)$ designs and $D(K_n, P_n, 1)$ designs for $n \equiv 2 \pmod 4$ remain open. We note that the existence problem of $D(K_{n,n}, P_{2n}, \lambda)$ designs also remains open.

[2] Dudeney posed the problem and he wrote "I discovered a subtle method for solving all cases" in his book ([2], p. 237), but he appears never to have published the method. No one knows whether he discoverd it, but at least he must have believed that there are solutions for all cases, so we may call it Dudeney's conjecture. One of the authors Nakamura also conjectured it and he wrote "it seems that a solution of the problem is possible for any number of persons" ([14], p. 218).

2 Notation and Preliminaries

We use the following notation to prove our theorems. For two sequences $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ of length n, define a sequence $X \times Y$ of length $2n$ as

$$X \times Y = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where x_i and y_i are variables.

Define s^jY $(0 \le j \le n-1)$ as

$$s^jY = (y_{n-j+1}, y_{n-j+2}, \ldots, y_{n-j}),$$

and define Y^{rev} and s^jY^{rev} $(0 \le j \le n-1)$ as

$$\begin{aligned} Y^{rev} &= (y_n, y_{n-1}, \ldots, y_1), \\ s^jY^{rev} &= (y_j, y_{j-1}, \ldots, y_{j+1}), \end{aligned}$$

where the subscripts of the y_i are calculated modulo n. Then we have

$$\begin{aligned} X \times s^jY &= (x_1, y_{n-j+1}, x_2, y_{n-j+2}, x_3, y_{n-j+3}, \ldots, x_n, y_{n-j}), \\ X \times s^jY^{rev} &= (x_1, y_j, x_2, y_{j-1}, x_3, y_{j-2}, \ldots, x_n, y_{j+1}). \end{aligned}$$

3 Proof of Theorem 1.1

Let n be an integer with $n \ge 2$. Let $K_{n,n} = (V, E)$ be the complete bipartite graph with the partite sets V_1 and V_2. Then we have $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$. Let $K_{V_1} = (V_1, E_1)$ and $K_{V_2} = (V_2, E_2)$ be the complete graphs on the vertex sets V_1 and V_2, respectively.

3.1 Proof of (1)

Let $n \ge 3$ be an odd and put $r = (n-1)/2$. Let $\mathcal{H} = \{H_i \mid 1 \le i \le r\}$ and $\mathcal{G} = \{G_i \mid 1 \le i \le r\}$ be Hamilton decompositions in K_{V_1} and K_{V_2}, respectively. (A *Hamilton decomposition* of a graph is a set of Hamilton cycles such that every edge of the graph appears exactly once. The complete graph K_{V_1} and K_{V_2} have Hamilton decompositions since n is odd.) Put $H_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$ and $G_i = (b_{1i}, b_{2i}, \ldots, b_{ni})$, where $a_{1i}, a_{2i}, \ldots, a_{ni} \in V_1$, $b_{1i}, b_{2i}, \ldots, b_{ni} \in V_2$, $1 \le i \le r$. Consider $H_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$ and $G_i = (b_{1i}, b_{2i}, \ldots, b_{ni})$ as sequences of vertices and put

$$\mathcal{D}_1 = \{H_i \times s^jG_i \mid 0 \le j \le n-1, 1 \le i \le r\}.$$

Consider $H_i \times s^jG_i$ as a Hamilton cycle in $K_{n,n}$. There are many representations of a Hamilton cycle, for example, $H_i = (a_{2i}, a_{3i}, \ldots, a_{ni}, a_{1i})$ or $H_i = (a_{ni}, a_{n-1,i}, \ldots, a_{2i}, a_{1i})$, etc, but $\mathcal{D}_1$ is uniquely determined as a set of Hamilton cycles. We will show that $\mathcal{D}_1$ is a $D(K_{n,n}, C_{2n}, 1)$ design.

Lemma 3.1 *Any 2-path in $K_{n,n}$ lies in exactly one cycle of $\mathcal{D}_1$.*

Proof. For any 2-path (x, y, z) with $x, z \in V_1, y \in V_2$, there is a Hamilton cycle H_t such that an edge $\{x, z\}$ belongs to H_t $(1 \le t \le r)$. Since y is one of the vertices in G_t, the 2-path (x, y, z) lies in a Hamilton cycle $H_t \times s^j G_t$ for some j $(0 \le j \le n-1)$.

For any 2-path (x, y, z) with $x, z \in V_2, y \in V_1$, there is a Hamilton cycle G_s such that an edge $\{x, z\}$ belongs to G_s $(1 \le s \le r)$. Since y is one of the vertices in H_s, the 2-path (x, y, z) lies in a Hamilton cycle $H_s \times s^k G_s$ for some k $(0 \le k \le n-1)$. Thus any 2-path in $K_{n,n}$ lies in a cycle of $\mathcal{D}_1$.

The number of the 2-paths in $K_{n,n}$ is $n^2(n-1)$, and the number of the 2-paths in a Hamilton cycle in $K_{n,n}$ is $2n$. Since the cardinality of $\mathcal{D}_1$ is nr, any 2-path in $K_{n,n}$ lies in exactly one cycle in $\mathcal{D}_1$. $\square$

From Lemma 3.1, $\mathcal{D}_1$ is a $D(K_{n,n}, C_{2n}, 1)$ design. This completes the proof of (1) of Theorem 1.1.

3.2 Proof of (3)

The proof of (3) is similar to that of (1). Let $n \ge 4$ be an even. Let $\mathcal{H} = \{H_i \mid 1 \le i \le n-1\}$ and $\mathcal{G} = \{G_i \mid 1 \le i \le n-1\}$ be Hamilton cycle double covers in K_{V_1} and K_{V_2}, respectively. (A *Hamilton cycle double cover* of a graph is a collection of Hamilton cycles such that every edge of the graph appears exactly twice. The complete graphs K_{V_1} and K_{V_2} have Hamilton cycle double covers.)

Put

$$\mathcal{D}_2 = \{H_i \times s^j G_i \mid 0 \le j \le n-1, 1 \le i \le n-1\}$$

similarly in **3.1**. Then $\mathcal{D}_2$ is a collection of Hamilton cycles in $K_{n,n}$.

Lemma 3.2 *Any 2-path in $K_{n,n}$ lies in exactly two cycles of $\mathcal{D}_2$.*

The proof of Lemma 3.2 is similar to that of Lemma 3.1, so we omit the proof here. From Lemma 3.2, $\mathcal{D}_2$ is a $D(K_{n,n}, C_{2n}, 2)$ design. When $n = 2$, it is easy to see that there is a $D(K_{n,n}, C_{2n}, 1)$ design. Thus there is a $D(K_{n,n}, C_{2n}, 2)$ design for every even $n \ge 2$.

This completes the proof of (3) of Theorem 1.1. We note that the assumption $n \equiv 2 \pmod 4$ is not required in the proof.

3.3 Proof of (2)

Let $n \geq 4$ be an integer with $n \equiv 0 \pmod 4$ and put $m = n/2$ and $l = n/4$. We put the vertex sets V_1 and V_2 as follows;

$$\begin{aligned} V_1 &= \{\infty, 0, 1, 2, \ldots, n-2\}, \\ V_2 &= \{\infty', 0', 1', 2', \ldots, (n-2)'\}. \end{aligned}$$

Addition of vertices except ∞, ∞' in V_1 and V_2 is calculated modulo $n-1$. Let σ be the vertex permutation $(\infty)(0\ 1\ 2\ \cdots\ n-2)(\infty')(0'\ 1'\ 2'\ \cdots\ (n-2)')$ in $K_{n,n}$ and put $\Sigma = \langle \sigma \rangle$. When $\mathcal{C}$ is a set of cycles in $K_{n,n}$, define

$$\Sigma\mathcal{C} = \{\sigma^t C \mid 0 \leq t \leq n-2,\ C \in \mathcal{C}\}.$$

For any integer i $(0 \leq i \leq n-2)$, define the 1-factors of K_{V_1} and K_{V_2}:

$$\begin{aligned} F_i &= \{\{\infty, i\}\} \cup \{\{a, b\} \in E_1 \mid a, b \neq \infty,\ a + b \equiv 2i \pmod{n-1}\}, \\ F'_i &= \{\{\infty', i'\}\} \cup \{\{a', b'\} \in E_2 \mid a', b' \neq \infty',\ a' + b' \equiv 2i \pmod{n-1}\}. \end{aligned}$$

Let $F_{0,m}$ and $F'_{0,m}$ denote the following sequences of vertices obtained from $F_0 \cup F_m$ and $F'_0 \cup F'_m$, respectively:

$$\begin{aligned} F_{0,m} = &\ (\infty, 0, 1, -1, 2, -2, 3, -3, \ldots, m-2, -(m-2), m-1, -(m-1)) \\ F'_{0,m} = &\ (0', 1', (-1)', 2', (-2)', 3', (-3)', \ldots, (m-2)', (-(m-2))', \\ &\qquad (m-1)', (-(m-1))', \infty'). \end{aligned}$$

Then we have

$$\begin{aligned} (F'_{0,m})^{rev} = &\ (\infty', (-(m-1))', (m-1)', (-(m-2))', (m-2)', \\ &\qquad \ldots, (-3)', 3', (-2)', 2', (-1)', 1', 0') \end{aligned}$$

and

$$\begin{aligned} F_{0,m} \times (F'_{0,m})^{rev} = &\ (\infty, \infty', 0, (-(m-1))', 1, (m-1)', -1, (-(m-2))', 2, (m-2)', \\ &\qquad \ldots, m-2, 2', -(m-2), (-1)', m-1, 1', -(m-1), 0'). \end{aligned}$$

Consider $F_{0,m} \times (F'_{0,m})^{rev}$ as a Hamilton cycle in $K_{n,n}$. Put

$$\mathcal{C} = \{F_{0,m} \times s^j (F'_{0,m})^{rev} \mid 0 \leq j \leq m-1\},$$

and rotate them, then we obtain a set of Hamilton cycles

$$\mathcal{D}_3 = \Sigma\mathcal{C}$$

in $K_{n,n}$. We will show that $\mathcal{D}_3$ is a $D(K_{n,n}, C_{2n}, 1)$ design.

Lemma 3.3 *Any* 2*-path in* $K_{n,n}$ *lies in a cycle in* $\mathcal{D}_3$.

Proof. If a 2-path (x, y', z) with $x, z \in V_1$ and $y' \in V_2$ lies in a cycle in $\mathcal{D}_3$, then the 2-path (x', y, z') with $x', z' \in V_2$ and $y \in V_1$ lies in a cycle in $\mathcal{D}_3$, and vice versa. Therefore we will only show that any 2-path (x, y', z) with $x, z \in V_1$ and $y' \in V_2$ lies in a cycle in $\mathcal{D}_3$.

For a 2-path (x, y', z) with $x, z \in V_1$ and $y' \in V_2$, there is a 2-path (a, b', c) with $\{a, c\} \in F_0$ and $b' \in V_2$ such that $(x, y', z) = \sigma^j(a, b', c)$ for some j $(0 \leq j \leq n-1)$. So we only need to show that any 2-path (a, b', c) with $\{a, c\} \in F_0$ and $b' \in V_2$ lies in a cycle in $\mathcal{D}_3$.

(i) A 2-path $(\infty, b', 0)$ lies in $\mathcal{C}$, where $b = \infty, 0, \pm 1, \pm 2, \ldots, \pm(l-1)$.

(ii) A 2-path $(a, b', -a)$ with $1 \leq a \leq l-1$ lies in $\mathcal{C}$, where $b = \pm(m-a), \pm(m-a+1), \ldots, \pm(m-1), \infty, 0, \pm 1, \pm 2, \ldots, \pm(l-1-a)$.

(iii) A 2-path $(a, b', -a)$ with $l \leq a \leq m-1$ lies in $\mathcal{C}$, where $b = \pm(m-a), \pm(m-a+1), \ldots, \pm(m+l-a-1)$.

(iv) A 2-path $(\infty, b', 0)$ lies in $\sigma^{m-1}\mathcal{C}$, where $b = \pm l, \pm(l+1), \ldots, \pm(m-1)$.

(v) A 2-path $(a, b', -a)$ with $1 \leq a \leq l-1$ lies in $\sigma^{m-1}\mathcal{C}$, where $b = \pm(l-a), \pm(l-a+1), \ldots, \pm(m-a-1)$.

(vi) A 2-path $(a, b', -a)$ with $a = l$ lies in $\sigma^{m-1}\mathcal{C}$, where $b = \infty, 0, \pm 1, \pm 2, \ldots, \pm(l-1)$.

(vii) A 2-path $(a, b', -a)$ with $l+1 \leq a \leq m-1$ lies in $\sigma^{m-1}\mathcal{C}$, where $b = \pm(m+l-a), \pm(m+l-a+1), \ldots, \pm(m-1), \infty, 0, \pm 1, \pm 2, \ldots, \pm(m-1-a)$.

Thus we complete the proof of Lemma 3.3. □

Lemma 3.4 *Any* 2*-path in* $K_{n,n}$ *lies in exactly one cycle in* $\mathcal{D}_3$.

Proof. The number of 2-paths in $K_{n,n}$ is $n^2(n-1)$, and the number of 2-paths in a Hamilton cycle in $K_{n,n}$ is $2n$. Since the cardinality of $\mathcal{D}_3$ is $n(n-1)/2$, any 2-path in $K_{n,n}$ lies in exactly one cycle in $\mathcal{D}_3$ by Lemma 3.3. □

From Lemma 3.4, $\mathcal{D}_3$ is a $D(K_{n,n}, C_{2n}, 1)$ design. This completes the proof of (2) of Theorem 1.1. This also completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We use the following notation to prove Theorem 1.2. Consider a sequence $Z = (z_1, z_2, \ldots, z_{2n})$ of length $2n$, where z_i $(1 \leq i \leq 2n)$ is a vertex in K_n. If there is an integer k $(1 \leq k \leq n-1)$ such that $z_{k-i} = z_{k+1+i}$ $(0 \leq i \leq n-1)$ and $z_{k+1}, z_{k+2}, \ldots, z_{k+n}$ are all different, where the subscripts of the z_i are calculated modulo $2n$, we call Z a *mirror sequence*. Then define $P(Z)$ as a path in K_n $(z_{k+1}, z_{k+2}, \ldots, z_{k+n})$ or $(z_{k+n+1}, z_{k+n+2}, \ldots, z_n, z_1, z_2, \ldots, z_k)$. Note that $(z_{k+1}, z_{k+2}, \ldots, z_{k+n})$ and $(z_{k+n+1}, z_{k+n+2}, \ldots, z_n, z_1, z_2, \ldots, z_k)$ are in reverse order, hence they are the same path.

4.1 Proof of (1)

It is trivial that there exists a $D(K_n, P_n, 1)$ design when $n = 3$. Then we assume that $n \geq 5$ is an odd integer. Let $\mathcal{H}$ be a Hamilton decomposition of K_n, and let $H = (a_1, a_2, \ldots, a_n)$ be a Hamilton cycle in $\mathcal{H}$. Consider $(a_1, a_2, \ldots, a_n)$ as a sequence of length n, then we have

$$H \times s^j H^{rev} = (a_1, a_j, a_2, a_{j-1}, a_3, a_{j-2}, \ldots, a_n, a_{j+1}),\ (0 \leq j \leq n-1).$$

Put $H \times s^j H^{rev} = (c_1, c_2, \ldots, c_{2n})$, then we have $c_k = c_{k+1}$ and $c_{k+n} = c_{k+n+1}$ for some k $(1 \leq k \leq n)$ (the subscripts of the c_i are calculated modulo $2n$). We see that $H \times s^j H^{rev}$ is a mirror sequence, and we have the Hamilton path

$$\begin{aligned} P(H \times s^j H^{rev}) &= (c_{k+1}, c_{k+2}, \ldots, c_{k+n}) \\ &= (c_{k+n+1}, c_{k+n+2}, \ldots, c_{2n}, c_1, c_2, \ldots, c_k). \end{aligned}$$

Put

$$\mathcal{P}(H) = \{P(H \times s^j H^{rev}) \mid 0 \leq j \leq n-1\},$$

then $\mathcal{P}(H)$ is a set of n Hamilton paths in K_n. The set $\mathcal{P}(H)$ is uniquely determined, i.e., it doesn't depend on the represetations of H.

Put

$$\mathcal{P}_1 = \bigcup_{H \in \mathcal{H}} \mathcal{P}(H),$$

then $\mathcal{P}_1$ is a set of $n(n-1)/2$ Hamilton paths in K_n.

Lemma 4.1 *$\mathcal{P}_1$ has each 2-path in K_n exactly once.*

Proof. Let (a, b, c) be any 2-path in K_n. The edge $\{a, c\}$ is contained in some Hamilton cycle H in $\mathcal{H}$, then we have $H = (a, c, \ldots)$. There is an integer j with $0 \leq j \leq n-1$ such that

$$H \times s^j H^{rev} = (a, b, c, \ldots, c, b, a),$$

that is, the Hamilton path $P(H \times s^j H^{rev})$ has the 2-path (a, b, c). Therefore the 2-path (a, b, c) belongs to $\mathcal{P}_1$ at least once.

Since the number of 2-paths in K_n is $n(n-1)(n-2)/2$ and the number of 2-paths in a Hamilton path in K_n is $n-2$, $\mathcal{P}_1$ has each 2-path in K_n exactly once. □

From Lemma 4.1, $\mathcal{P}_1$ is a $D(K_n, P_n, 1)$ design. This completes the proof of (1) of Theorem 1.2.

4.2 Proof of (3)

Let $n \geq 4$ be even and $\mathcal{H}$ a Hamilton cycle double cover of K_n. The proof of (3) is similar to that of (1), so it is omitted. We obtain a $D(K_n, P_n, 2)$ design $\mathcal{P}_2$ and we complete the proof of (3) of Theorem 1.2. Note that the assumption $n \equiv 2 \pmod 4$ is not required in the proof.

4.3 Proof of (2)

Let $n \geq 4$ be even with $n \equiv 0 \pmod 4$ and put $m = n/2$ and $l = n/4$. We use the same notation in **3.3**. $K_{n,n} = (V, E)$ is the complete bipartite graph with $V = V_1 \cup V_2$. $K_{V_1} = (V_1, E_1)$ and $K_{V_2} = (V_2, E_2)$ are the complete graphs on the vertex sets

$$\begin{aligned} &V_1 = \{\infty, 0, 1, 2, \ldots, n-2\} \text{ and} \\ &V_2 = \{\infty', 0', 1', 2', \ldots, (n-2)'\}, \end{aligned}$$

respectively.

We will construct a $D(K_n, P_n, 1)$ design on the complete graph K_{V_1}. Define a sequence of vertices in K_{V_1} as follows:

$$\begin{aligned} [F_{0,m} \times (F'_{0,m})^{rev}]_{K_{V_1}} = \; & (\infty, \infty, 0, -(m-1), 1, m-1, -1, -(m-2), 2, m-2, \\ & \ldots, -l, l, l, -l, \\ & \ldots, m-2, 2, -(m-2), -1, m-1, 1, -(m-1), 0). \end{aligned}$$

This is the sequence obtained from $F_{0,m} \times (F'_{0,m})^{rev}$ by deleting the prime symbol ($'$). It is a mirror sequence and we have the Hamilton path in K_{V_1},

$$\begin{aligned} P([F_{0,m} \times (F'_{0,m})^{rev}]_{K_{V_1}}) = \; & (\infty, 0, -(m-1), 1, m-1, -1, -(m-2), 2, m-2, \\ & \ldots, -l, l). \end{aligned}$$

Put

$$\mathcal{P} = \{P([F_{0,m} \times s^j (F'_{0,m})^{rev}]_{K_{V_1}}) \mid 0 \leq j \leq m-1\},$$

and rotate them, then we obtain a set of Hamilton paths

$$\mathcal{P}_3 = \Sigma_1 \mathcal{P}$$

in K_{V_1}, where Σ_1 denotes the group generated by the vertex permutation $(\infty)(0\ 1\ 2\ \cdots\ n-2)$.

We will show that $\mathcal{P}_3$ is a $D(K_n, P_n, 1)$ design. Note that a 2-path (x, y, z) belongs to $\mathcal{P}_3$ if and only if 2-paths (x, y', z) and/or (x', y, z') belong to $\mathcal{D}_3$ defined in **3.3**, where $x, y, z \in V_1$, $x', y', z' \in V_2$, and x, y, z are all different.

Lemma 4.2 *$\mathcal{P}_3$ has each 2-path in K_{V_1} exactly once.*

Proof. Let (x, y, z) be any 2-path in K_{V_1}. From Lemma 3.3, the 2-path (x, y', z) with $x, z \in V_1$ and $y' \in V_2$ lies in a cycle in $\mathcal{D}_3$. So the 2-path (x, y, z) lies in a path in $\mathcal{P}_3$.

The number of 2-paths in K_{V_1} is $n(n-1)(n-2)/2$ and the number of 2-paths in a Hamilton path in K_{V_1} is $n-2$. Since the cardinality of $\mathcal{P}_3$ is $n(n-1)/2$, any 2-path in K_{V_1} lies in exactly one cycle in $\mathcal{P}_3$. □

From Lemma 4.2, $\mathcal{P}_3$ is a $D(K_n, P_n, 1)$ design. This completes the proof of (2) of Theorem 1.2. This also completes the proof of Theorem 1.2.

5 Remarks

Let $n \geq 3$ be an integer. We obtain a $D(K_n, P_n, \lambda)$ design by deleting a vertex of a $D(K_{n+1}, C_{n+1}, \lambda)$ design. Therefore Theorem 1.2 (1) and (3) follow from Theorem A; however, in this paper, we proved Theorem 1.2 (1) and (3) without Theorem A. In fact, the proof of Theorem A is long and complicated so a proof not using Theorem A would be desirable.

Theorem 1.1 (1) and (3) also follow from Theorem A since we have the following proposition.

Proposition 5.1 *Let $n \geq 2$ be an integer. If there exists a $D(K_{n+1}, C_{n+1}, \lambda)$ design, then there exists a $D(K_{n,n}, C_{2n}, \lambda)$ design.*

Proof. Let $V_1 \cup V_2$ be the vertex set of $K_{n,n}$, where $V_1 = \{1, 2, \dots, n\}$, $V_2 = \{1', 2', \dots, n'\}$, and let $V = \{\infty\} \cup V_1$ be the vertex set of K_{n+1}. Assume that there exists a $D(K_{n+1}, C_{n+1}, \lambda)$ design $\mathcal{D}$.

For a Hamilton cycle $H = (\infty, a_1, a_2, \dots, a_n)$ in $\mathcal{D}$, define a Hamilton cycle $f(H)$ in $K_{n,n}$ as follows:

$$f(H) = \begin{cases} (a_1, a_2', \dots, a_{n-1}, a_n', a_n, a_{n-1}', \dots, a_2, a_1') & (n \text{ is even}) \\ (a_1, a_2', \dots, a_{n-1}', a_n, a_n', a_{n-1}, \dots, a_2, a_1') & (n \text{ is odd}). \end{cases}$$

Note that $f(H)$ is well-defined, i.e., $f((\infty, a_1, a_2, \dots, a_n)) = f((\infty, a_n, a_{n-1}, \dots, a_1))$.

Put

$$\mathcal{D}^* = \{f(H) \mid H \in \mathcal{D}\},$$

and we show that $\mathcal{D}^*$ is a $D(K_{n,n}, C_{2n}, \lambda)$ design.

For any 2-paths (a, b', c) and (a', b, c') in $K_{n,n}$ such that a, b, c are all different, there exist λ Hamilton cycles H_i $(1 \leq i \leq \lambda)$ in $\mathcal{D}$ which have a 2-path (a, b, c). Then each $f(H_i)$ contains the 2-paths (a, b', c) and (a', b, c') $(1 \leq i \leq \lambda)$.

For any 2-paths (a, a', b) and (a', a, b') in $K_{n,n}$ with $a \neq b$, there exist λ Hamilton cycles H_i $(1 \leq i \leq \lambda)$ in $\mathcal{D}$ which have a 2-path (∞, a, b). Then we have $f(H_i) =$

$(a, b', \ldots, b, a')$, so each $f(H_i)$ contains the 2-paths (a, a', b) and (a', a, b') $(1 \leq i \leq \lambda)$. Hence $\mathcal{D}^*$ contains each 2-path in $K_{n,n}$ at least λ times.

The set $\mathcal{D}^*$ contains $\lambda n(n-1)/2$ Hamilton cycles, and a Hamilton cycle contains $2n$ 2-paths. The number of 2-paths in $K_{n,n}$ is $n^2(n-1)$, so we see that $\mathcal{D}^*$ contains each 2-path in $K_{n,n}$ exactly λ times. Therefore $\mathcal{D}^*$ is a $D(K_{n,n}, C_{2n}, \lambda)$ design. $\square$

As stated in Section 1, the problem of constructing $D(K_n, C_n, 1)$ designs for all n, i.e., Dudeney's round table problem, is still open. If the problem is solved, then the problems of constructing $D(K_{n,n}, C_{2n}, 1)$ designs and $D(K_n, P_n, 1)$ designs are solved. In this sense, the problem of constructing $D(K_n, C_n, 1)$ designs for all n would be an interesting open problem.

References

[1] J. Akiyama, M. Kobayashi and G. Nakamura, Uniform coverings of 2-paths with 6-paths in the complete graph, J. Akiyama et al. (Eds.): Combinatorial Geometry and Graph Theory, IJCCGGT 2003, Lecture Notes in Computer Science, Springer-Verlag, 3330 (2005) 25-33.

[2] H. E. Dudeney, "The Canterbury Puzzles", W. Heinemann, London, 1907; Dover, New York, 2002.

[3] H. E. Dudeney, "Amusements in Mathematics", Thomas Nelson and Sons, 1917; Dover, New York, 1970.

[4] K. Heinrich, D. Langdeau and H. Verrall, Covering 2-paths uniformly, *J. Combin. Des.* **8** (2000) 100-121.

[5] K. Heinrich and G. Nonay, Exact coverings of 2-paths by 4-cycles, *J. Combin. Theory* (A) **45** (1987) 50-61.

[6] M. Kobayashi, Kiyasu-Z. and G. Nakamura, A solution of Dudeney's round table problem for an even number of people, *J. Combin. Theory* (A) **63** (1993) 26-42.

[7] M. Kobayashi, B. D. McKay, N. Mutoh and G. Nakamura, Black 1-factors and Dudeney sets, *J. Combin. Math. Combin. Comput.*, **75** (2010) 167-174.

[8] M. Kobayashi, N. Mutoh, Kiyasu-Z. and G. Nakamura, Double coverings of 2-paths by Hamilton cycles, *J. Combin. Designs* **10** (2002) 195-206.

[9] M. Kobayashi and G. Nakamura, Resolvable coverings of 2-paths by 4-cycles, *J. Combinatorial Theory* (A) **60** (1992) 295-297.

[10] M. Kobayashi and G. Nakamura, Uniform coverings of 2-paths by 4-paths, *Australas. J. Combin.* **24** (2001) 301-304.

[11] M. Kobayashi and G. Nakamura, Uniform coverings of 2-paths with 6-cycles in the complete graph, *Australas. J. Combin.* **34** (2006) 299-304.

[12] M. Kobayashi, G. Nakamura and C. Nara, Uniform coverings of 2-paths with 5-paths in K_{2n}, *Australas. J. Combin.* **27** (2003) 247-252.

[13] M. Kobayashi, G. Nakamura and C. Nara, Uniform coverings of 2-paths with 5-paths in the complete graph, *Discrete Mathematics* **299** (2005) 154-161.

[14] The 13 knights by Victor Meally, Dublin, Ireland, in "Solutions to problems, conjectures, and alphametics", *J. Recreational Mathematics*, **9** (3) (1976-77), 216-218.