Uniform coverings of 2-paths in the complete bipartite directed graph

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Abstract Let G be a directed graph and H a subgraph of G. A $D(G,H,\lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every directed 2-path in G lies in exactly λ subgraphs in $\mathcal{D}$. In this paper, we show that there exists a $D(K^*_{n,n}, \overrightarrow{C}_{2n}, 1)$ design for every $n \geq 2$, where $K^*_{n,n}$ is the complete bipartite directed graph and $\overrightarrow{C}_{2n}$ is a directed Hamilton cycle in $K^*_{n,n}$.

1 Introduction

Consider a graph G and a subgraph H of G. A $D(G,H,\lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every 2-path (path of length 2) lies in exactly λ subgraphs in $\mathcal{D}$. Analogously, when G is a directed graph (digraph) and H is a subgraph of G (a subgraph of a directed graph means a directed subgraph), a $D(G,H,\lambda)$ *design* is a multiset $\mathcal{D}$ of subgraphs of G each isomorphic to H so that every directed 2-path lies in exactly λ subgraphs in $\mathcal{D}$.[1] We call these designs *Dudeney designs*.

The following notation will be used. K_n is the complete graph on n vertices, $K_{n,n}$ is the complete bipartite graph on partite sets with n vertices each, C_k is a cycle on k vertices, and P_k is a path on k vertices. K^*_n is the complete digraph on n vertices and $K^*_{n,n}$ is the complete bipartite digraph on partite sets with n vertices each. K^*_n and $K^*_{n,n}$ are digraphs which are obtained from K_n and $K_{n,n}$, respectively, by substituting oppositely directed edges for each edge. $\overrightarrow{C}_k$ is a directed cycle on k vertices and $\overrightarrow{P}_k$ is a directed path on k vertices.

We restrict our attention to $D(G,H,\lambda)$ designs in which G is K_n, $K_{n,n}$, K^*_n or $K^*_{n,n}$, and H is a Hamilton cycle or a Hamilton path. (In the case of k-cycles and k-paths, see [5].) A $D(K_n, C_n, 1)$ design is a solution of the famous *Dudeney's round table problem* which asks for a seating of n people at a round table on consecutive days so that each person sat between every pair of other people exactly once [3].

The following theorems are known.

Theorem A [2, 4] *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n, C_n, 1)$ design when n is even.*
(2) *There exists a $D(K_n, C_n, 2)$ design when n is odd.*

[1] For more general definition, see [1].

Theorem B [5] *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n, P_n, 1)$ design when $n \equiv 0, 1, 3 \pmod 4$.*
(2) *There exists a $D(K_n, P_n, 2)$ design when $n \equiv 2 \pmod 4$.*

Theorem C [5] *Let $n \geq 2$ be an integer.*
(1) *There exists a $D(K_{n,n}, C_{2n}, 1)$ design when $n \equiv 0, 1, 3 \pmod 4$.*
(2) *There exists a $D(K_{n,n}, C_{2n}, 2)$ design when $n \equiv 2 \pmod 4$.*

For the complete digraph K_n^* and the complete bipartite digraph $K_{n,n}^*$, we obtain the following corollaries immediately from the above theorems.

Corollary A′ *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n^*, \overrightarrow{C}_n, 1)$ design when n is even.*
(2) *There exists a $D(K_n^*, \overrightarrow{C}_n, 2)$ design when n is odd.*

Corollary B′ *Let $n \geq 3$ be an integer.*
(1) *There exists a $D(K_n^*, \overrightarrow{P}_n, 1)$ design when $n \equiv 0, 1, 3 \pmod 4$.*
(2) *There exists a $D(K_n^*, \overrightarrow{P}_n, 2)$ design when $n \equiv 2 \pmod 4$.*

Corollary C′ *Let $n \geq 2$ be an integer.*
(1) *There exists a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design when $n \equiv 0, 1, 3 \pmod 4$.*
(2) *There exists a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 2)$ design when $n \equiv 2 \pmod 4$.*

We note that the $D(K_n, C_n, 2)$ design (n is odd) constructed in [4] does not induce a $D(K_n^*, \overrightarrow{C}_n, 1)$ design, and the $D(K_n, P_n, 2)$ design and the $D(K_{n,n}, C_{2n}, 2)$ design ($n \equiv 2 \pmod 4$) constructed in [5] does not induce a $D(K_n^*, \overrightarrow{P}_n, 1)$ design and a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design, respectively.

In this paper, we obtain the following theorem.

Theorem 1.1 *There exists a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design for every $n \geq 2$.*

The method of the proof is similar to the method used in [5]. For two sequences $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ of length n, define a sequence $X \times Y$ of length $2n$ as

$$X \times Y = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

Define $s^j Y$ $(0 \leq j \leq n-1)$ as

$$s^j Y = (y_{n-j+1}, y_{n-j+2}, \ldots, y_{n-j}),$$

where the subscripts of the y_i are calculated modulo n. Then we have

$$X \times s^j Y = (x_1, y_{n-j+1}, x_2, y_{n-j+2}, x_3, y_{n-j+3}, \ldots, x_n, y_{n-j}).$$

2 Proof of Theorem 1.1

A *Hamilton decomposition* $\mathcal{H}$ of a digraph is a set of Hamilton cycles such that every directed edge (arc) of the digraph appears in $\mathcal{H}$ exactly once. Note that a Hamilton cycle in a digraph means a directed Hamilton cycle.

Proposition 2.1 *Let $n \geq 3$ be an integer. If there exists a Hamilton decomposition of K_n^*, then there exists a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design.*

Proof. Let $V = V_1 \cup V_2$ be the vertex set of $K_{n,n}^*$, where V_1 and V_2 are the partite sets with $|V_1| = |V_2| = n$. Let $K_{V_1}^*$ and $K_{V_2}^*$ be the complete digraphs on the vertex sets V_1 and V_2, respectively.

Let $\mathcal{H} = \{H_i \mid 1 \leq i \leq n-1\}$ and $\mathcal{G} = \{G_i \mid 1 \leq i \leq n-1\}$ be Hamilton decompositions of $K_{V_1}^*$ and $K_{V_2}^*$, respectively. Put $H_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$ and $G_i = (b_{1i}, b_{2i}, \ldots, b_{ni})$, where $a_{1i}, a_{2i}, \ldots, a_{ni} \in V_1$, $b_{1i}, b_{2i}, \ldots, b_{ni} \in V_2$ $(1 \leq i \leq n-1)$. Consider $H_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$ and $G_i = (b_{1i}, b_{2i}, \ldots, b_{ni})$ as sequences of vertices and put

$$\mathcal{D}_1 = \{H_i \times s^j G_i \mid 0 \leq j \leq n-1, 1 \leq i \leq n-1\}.$$

Consider $H_i \times s^j G_i$ as a Hamilton cycle in $K_{n,n}^*$. There are many representations of a Hamilton cycle, for example, $H_i = (a_{2i}, a_{3i}, \ldots, a_{ni}, a_{1i})$ or $H_i = (a_{3i}, a_{4i}, \ldots, a_{1i}, a_{2i})$, etc, but $\mathcal{D}_1$ is uniquely determined.

We will show that any directed 2-path in $K_{n,n}^*$ lies in exactly one cycle of $\mathcal{D}_1$.

For a directed 2-path (x, y, z) with $x, z \in V_1, y \in V_2$, there is a Hamilton cycle H_t such that a directed edge (x, z) belongs to H_t $(1 \leq t \leq n-1)$. Since y is one of the vertices in G_t, the directed 2-path (x, y, z) lies in a Hamilton cycle $H_t \times s^j G_t$ for some j $(0 \leq j \leq n-1)$.

For a directed 2-path (x, y, z) with $x, z \in V_2, y \in V_1$, there is a Hamilton cycle G_s such that a directed edge (x, z) belongs to G_s $(1 \leq s \leq n-1)$. Since y is one of the vertices in H_s, the directed 2-path (x, y, z) lies in a Hamilton cycle $H_s \times s^k G_s$ for some k $(0 \leq k \leq n-1)$. Thus any 2-path in $K_{n,n}^*$ lies in a cycle of $\mathcal{D}_1$.

The number of directed 2-paths in $K_{n,n}^*$ is $2n^2(n-1)$, and the number of directed 2-paths in a Hamilton cycle in $K_{n,n}^*$ is $2n$. Since the cardinality of $\mathcal{D}_1$ is $n(n-1)$, any directed 2-path in $K_{n,n}^*$ lies in exactly one cycle in $\mathcal{D}_1$.

Therefore $\mathcal{D}_1$ is $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design, which completes the proof. □

When $n \geq 8$, there is a Hamilton decomposition of the complete digraph K_n^* [6]. When $n = 3, 5, 7$, there is a Hamilton decomposition of K_n^* since there is a Hamilton decomposition of K_n. Therefore when $n = 3, 5$ and $n \geq 7$, there is a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design from Prop. 2.1. When $n = 2$, it is trivial that there is a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design. When $n = 4$, there is a $D(K_{n,n}^*, \overrightarrow{C}_{2n}, 1)$ design from Cor. C′. When $n = 6$, there is a $D(K^*_{n,n}, \overrightarrow{C}_{2n}, 1)$ design as we found a $D(K_{n,n}, C_{2n}, 1)$ design[2] with the aid of a computer. Thus there is a $D(K^*_{n,n}, \overrightarrow{C}_{2n}, 1)$ design for every $n \geq 2$.

This completes the proof of Theorem 1.1.

3 Open problems

The existence problem of $D(K^*_{n,n}, \overrightarrow{C}_{2n}, 1)$ designs was solved in this paper. For $D(G, H, \lambda)$ designs in which G is K_n, $K_{n,n}$, K^*_n or $K^*_{n,n}$, and H is a Hamilton cycle or a Hamilton path, the remaining problems are whether the following designs exist:
1. $D(K_n, C_n, 1)$ designs for odd n,
2. $D(K_n, P_n, 1)$ designs for n with $n \equiv 2 \pmod 4$,
3. $D(K_{n,n}, C_{2n}, 1)$ designs for n with $n \equiv 2 \pmod 4$,
4. $D(K_{n,n}, P_{2n}, \lambda)$ designs for n and λ,
5. $D(K^*_n, \overrightarrow{C}_n, 1)$ designs for odd n,
6. $D(K^*_n, \overrightarrow{P}_n, 1)$ designs for n with $n \equiv 2 \pmod 4$,
7. $D(K^*_{n,n}, \overrightarrow{P}_{2n}, \lambda)$ designs for n and λ.

If the designs 1 exist, then the designs 2, 3 and 5 exist[3] and if the designs 2 exist, then the designs 6 exist. In this sense, a $D(K_n, C_n, 1)$ design, i.e., a solution of Dudeney's round table problem, would be important among them.

References

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[2] $\{(0,0',1,1',2,2',3,3',4,4',5,5')$, $(0,0',2,1',3,2',1,3',5,5',4,4')$, $(0,0',3,1',1,2',4,5',2,4',5,3')$, $(0,0',4,1',1,3',2,4',3,5',5,2')$, $(0,0',5,2',1,5',2,3',3,4',4,1')$, $(0,1',1,4',4,2',2,3',5,0',3,5')$, $(0,1',2,3',4,5',1,0',5,4',3,2')$, $(0,1',3,5',2,0',1,3',4,2',5,4')$, $(0,1',5,4',1,5',3,2',4,0',2,3')$, $(0,2',1,4',2,1',5,0',4,5',3,3')$, $(0,2',2,4',4,0',3,3',5,1',1,5')$, $(0,2',4,1',3,3',1,5',5,0',2,4')$, $(0,3',1,4',3,0',2,2',5,1',4,5')$, $(0,3',4,0',1,2',2,5',5,1',3,4')$, $(0,4',1,0',3,2',5,3',4,1',2,5')\}$, where $\{0,1,2,3,4,5\} \cup \{0',1',2',3',4',5'\}$ is the vertex set of $K_{n,n}$.

[3] The designs 1 induce the designs 2 by deleting a vertex. For the fact that the designs 1 induce the designs 3, see Prop. 5.1 in [5].

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